Random Numbers

- Randomness
- Applications in scientific computing
- Random variables and Pseudo Random Numbers
- Properties of a good random number generator (RNG)
- RNGs
  - Mid-square method
  - Linear Congruential Generator
  - Linear Shift Feedback Register
- Quasi Random Numbers

Outline

- Random number generation
- Examples and limitations
- Tests for RNGs
- Description of practical exercise

Random Number Generation

Desirable Attributes:

- Uniformity
- Independence
- Efficiency
- Replicability
- Long Cycle Length

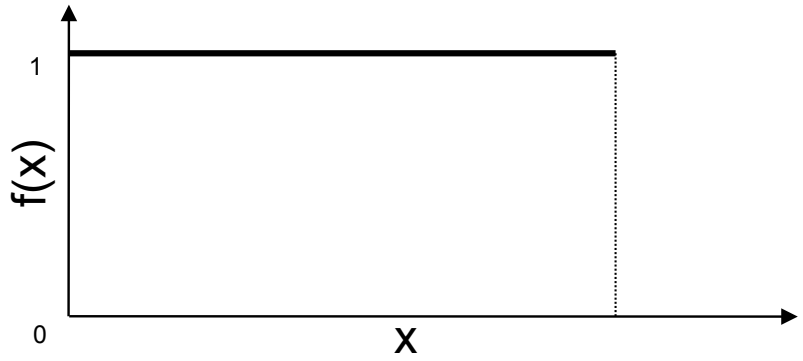
Random Number Generation (cont.)

Each random number R_t is an independent sample drawn from a continuous uniform distribution between 0 and 1

$$\text{pdf: } f(x) = \begin{cases} 1 , 0 \le x \le 1 \\ 0 , \text{otherwise} \end{cases}$$

Random Number Generation (cont.)

PDF:



$$E(R) = \int_0^1 x dx = [x^2/2]_0^1 = 1/2$$

$$V(R) = \int_0^1 x^2 dx - [E(R)]^2$$

$$= [x^3/3]_0^1 - (1/2)^2 = 1/3 - 1/4$$

$$= 1/12$$

Techniques for Generating Random Number

MidSquare

Example:

$X_0 = 7182$ (seed)

$X_0^2 = 51581124$

==> $R_1 = 0.5811$

$X_0^2 = (5811)^2 = 33767721$

==> $R_2 = 0.7677$

etc.

Techniques for Generating Random Number (cont.)

Note: Cannot choose a seed that guarantees that the sequence will not degenerate and will have a long period. Also, zeros, once they appear, are carried in subsequent numbers.

Ex1: $X_0 = 5197$ (seed) $X_0^2 = 27\underline{0088}09$

==> $R_1 = 0.0088$ $X_1^2 = 00\underline{0077}44$

==> $R_2 = 0.0077$

Ex2: $X_0 = 4500$ (seed) $X_0^2 = 20\underline{2500}00$

==> $R_1 = 0.2500$ $X_1^2 = 06\underline{2500}00$

==> $R_2 = 0.2500$

Techniques for Generating Random Number (cont.)

- **Multiplicative Congruential Method:**

  Basic Relationship

  $X_{i+1} = a X_i \pmod m$, where $a \geq 0$ and $m \geq 0$

  Most natural choice for m is one that equals to the capacity of a computer word.

  $m = 2^b$ (binary machine), where b is the number of bits in the computer word.

  $m = 10^d$ (decimal machine), where d is the number of digits in the computer word.

Techniques for Generating Random Number (cont.)

The max period(P) is:

- For m a power of 2, say $m = 2^b$, and $c \neq 0$, the longest possible period is $P = m = 2^b$, which is achieved provided that c is relatively prime to m (that is, the greatest common factor of c and m is 1), and $a = 1 + 4k$, where k is an integer.
- For m a power of 2, say $m = 2^b$, and $c = 0$, the longest possible period is $P = m / 4 = 2^{b-2}$, which is achieved provided that the seed X_0 is odd and the multiplier, a, is given by $a = 3 + 8k$ or $a = 5 + 8k$, for some $k = 0, 1, ...$

Techniques for Generating Random Number (cont.)

- For **m** a prime number and **c = 0**, the longest possible period is **P = m - 1**, which is achieved provided that the multiplier, **a**, has the property that the smallest integer k such that $a^k - 1$ is divisible by m is k = m - 1,

Techniques for Generating Random Number (cont.)

(Example)

Using the multiplicative congruential method, find the period of the generator for a = 13, m = 2^6, and X_0 = 1, 2, 3, and 4. The solution is given in next slide. When the seed is 1 and 3, the sequence has period 16. However, a period of length eight is achieved when the seed is 2 and a period of length four occurs when the seed is 4.

Techniques for Generating Random Number (cont.)

Period Determination Using Various seeds

i	X_i	X_i	X_i	X_i
0	1	2	3	4
1	13	26	39	52
2	41	18	59	36
3	21	42	63	20
4	17	34	51	4
5	29	58	23	
6	57	50	43	
7	37	10	47	
8	33	2	35	
9	45		7	
10	9		27	
11	53		31	
12	49		19	
13	61		55	
14	25		11	
15	5		15	
16	1		3	

Techniques for Generating Random Number (cont.)

- Linear Congruential Method:

  $X_{i+1} = (aX_i + c) \bmod m, i = 0, 1, 2....$

  (Example)

  let $X_0 = 27$, $a = 17$, $c = 43$, and $m = 100$, then

  $X_1 = (17*27 + 43) \bmod 100 = 2$

  $R_1 = 2 / 100 = 0.02$

  $X_2 = (17*2 + 43) \bmod 100 = 77$

  $R_2 = 77 / 100 = 0.77$

  .....

Attributes of RNGs

1. Uniformity
2. Independence
3. Efficiency
4. Replicability
5. Long Cycle Length

Test for Random Numbers

1. ***Frequency test***. Uses the Kolmogorov-Smirnov or the chi-square test to compare the distribution of the set of numbers generated to a uniform distribution.
2. ***Runs test***. Tests the runs up and down or the runs above and below the mean by comparing the actual values to expected values. The statistic for comparison is the chi-square.
3. ***Autocorrelation test***. Tests the correlation between numbers and compares the sample correlation to the expected correlation of zero.

Test for Random Numbers (cont.)

4. ***Gap test***. Counts the number of digits that appear between repetitions of a particular digit and then uses the Kolmogorov-Smirnov test to compare with the expected number of gaps.
5. ***Poker test***. Treats numbers grouped together as a poker hand. Then the hands obtained are compared to what is expected using the chi-square test.

Steps to Perform a Test of Hypothesis

- State the null and alternative hypotheses
- Select the distribution to use
- Determine the rejection and non-rejection regions
- Calculate the value of the test statistic
- Make a decision

Test for Random Numbers (cont.)

In testing for uniformity, the hypotheses are as follows:

H_0: $R_i \sim U[0,1]$

H_1: $R_i \neq U[0,1]$

The null hypothesis, H_0, reads that the numbers are distributed uniformly on the interval [0,1].

Test for Random Numbers (cont.)

In testing for independence, the hypotheses are as follows;

H_0: R_i ~ independently

H_1: R_i ≠ independently

This null hypothesis, H_0, reads that the numbers are independent. Failure to reject the null hypothesis means that no evidence of dependence has been detected on the basis of this test. This does not imply that further testing of the generator for independence is unnecessary.

Test for Random Numbers (cont.)

- Level of significance α

  $\alpha = P(\text{reject } H_0 \mid H_0 \text{ true})$

  Frequently, α is set to 0.01 or 0.05

(Hypothesis)

	Actually True	Actually False
Accept	$1 - \alpha$	β (Type II error)
Reject	α (Type I error)	$1 - \beta$

Test for Random Numbers (cont.)

- The *Gap Test* measures the number of digits between successive occurrences of the same digit.

(Example) length of gaps associated with the digit 3.

4, 1, 3, 5, 1, 7, 2, 8, 2, 0, 7, 9, 1, 3, 5, 2, 7, 9, 4, 1, 6, 3
3, 9, 6, 3, 4, 8, 2, 3, 1, 9, 4, 4, 6, 8, 4, 1, 3, 8, 9, 5, 5, 7
3, 9, 5, 9, 8, 5, 3, 2, 2, 3, 7, 4, 7, 0, 3, 6, 3, 5, 9, 9, 5, 5
5, 0, 4, 6, 8, 0, 4, 7, 0, 3, 3, 0, 9, 5, 7, 9, 5, 1, 6, 6, 3, 8
8, 8, 9, 2, 9, 1, 8, 5, 4, 4, 5, 0, 2, 3, 9, 7, 1, 2, 0, 3, 6, 3

Note: eighteen 3's in list

==> 17 gaps, the first gap is of length 10

Test for Random Numbers (cont.)

We are interested in the frequency of gaps.

P(gap of 10) = P(not 3) ···P(not 3) P(3) , note: there are 10 terms of the type P(not 3)

$$= (0.9)^{10}\,(0.1)$$

The theoretical frequency distribution for randomly ordered digit is given by

$$F(x) = 0.1 \sum_{n=0}^{x} (0.9)^n = 1 - 0.9^{x+1}$$

Note: observed frequencies for all digits are compared to the theoretical frequency using the Kolmogorov-Smirnov test.

Test for Random Numbers (cont.)

(Example)

Based on the frequency with which gaps occur, analyze the 110 digits above to test whether they are independent. Use $\alpha = 0.05$. The number of gaps is given by the number of digits minus 10, or 100. The number of gaps associated with the various digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
# of Gaps	7	8	8	17	10	13	7	8	9	13

Test for Random Numbers (cont.)

Gap Test Example

Gap Length	Frequency	Frequency	Relative Frequency	Cum. Relative F(x)	$\lvert F(x) - S_N(x) \rvert$
0-3	35	0.35	0.35	0.3439	0.0061
4-7	22	0.22	0.57	0.5695	0.0005
8-11	17	0.17	0.74	0.7176	0.0224
12-15	9	0.09	0.83	0.8147	0.0153
16-19	5	0.05	0.88	0.8784	0.0016
20-23	6	0.06	0.94	0.9202	0.0198
24-27	3	0.03	0.97	0.9497	0.0223
28-31	0	0.00	0.97	0.9657	0.0043
32-35	0	0.00	0.97	0.9775	0.0075
36-39	2	0.02	0.99	0.9852	0.0043
40-43	0	0.00	0.99	0.9903	0.0003
44-47	1	0.01	1.00	0.9936	0.0064

Test for Random Numbers (cont.)

The critical value of D is given by

$$D_{0.05} = 1.36 / \sqrt{100} = 0.136$$

Since $D = \max |F(x) - S_N(x)| = 0.0224$ is less than $D_{0.05}$, do not reject the hypothesis of independence on the basis of this test.

Test for Random Numbers (cont.)

- *Run Tests* (Up and Down)

  Consider the 40 numbers; both the Kolmogorov-Smirnov and Chi-square would indicate that the numbers are uniformly distributed. But, not so.

0.08	0.09	0.23	0.29	0.42	0.55	0.58	0.72	0.89	0.91
0.11	0.16	0.18	0.31	0.41	0.53	0.71	0.73	0.74	0.84
0.02	0.09	0.30	0.32	0.45	0.47	0.69	0.74	0.91	0.95
0.12	0.13	0.29	0.36	0.38	0.54	0.68	0.86	0.88	0.91

Test for Random Numbers (cont.)

Now, rearrange and there is less reason to doubt independence.

0.41	0.68	0.89	0.84	0.74	0.91	0.55	0.71	0.36	0.30
0.09	0.72	0.86	0.08	0.54	0.02	0.11	0.29	0.16	0.18
0.88	0.91	0.95	0.69	0.09	0.38	0.23	0.32	0.91	0.53
0.31	0.42	0.73	0.12	0.74	0.45	0.13	0.47	0.58	0.29

Test for Random Numbers (cont.)

Concerns:

- Number of runs
- Length of runs

Note: If N is the number of numbers in a sequence, the maximum number of runs is N-1, and the minimum number of runs is one.

If "a" is the total number of runs in a sequence, the mean and variance of "a" is given by

Test for Random Numbers (cont.)

$\mu_a = (2n - 1) / 3$

$\sigma_a^2 = (16N - 29) / 90$

For $N > 20$, the distribution of "a" approximated by a normal distribution, $N(\mu_a , \sigma_a^2)$.

This approximation can be used to test the independence of numbers from a generator.

$$Z_0 = (a - \mu_a) / \sigma_a$$

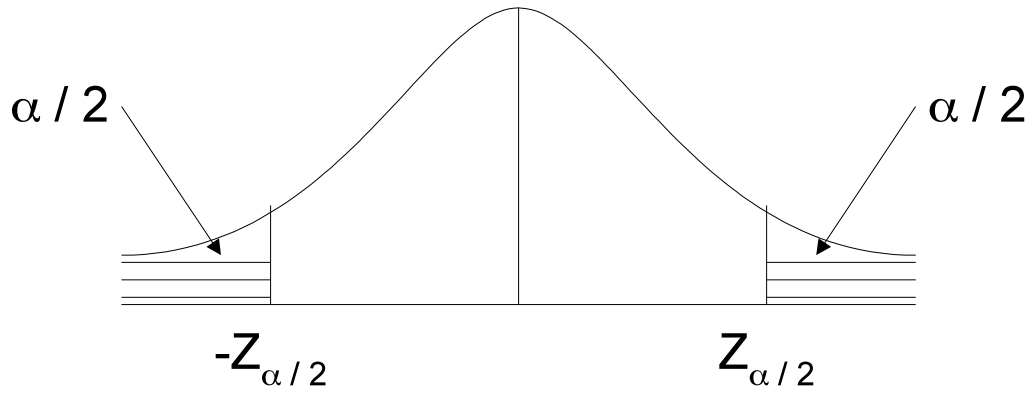
Test for Random Numbers (cont.)

Substituting for μ_a and σ_a ==>

$$Z_a = \{a - [(2N-1)/3]\} / \{\sqrt{(16N-29)/90}\},$$

where $Z \sim N(0,1)$

Acceptance region for hypothesis of independence $-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$



Test for Random Numbers (cont.)

(Example)

Based on runs up and runs down, determine whether the following sequence of 40 numbers is such that the hypothesis of independence can be rejected where $\alpha = 0.05$.

0.41	0.68	0.89	0.94	0.74	0.91	0.55	0.62	0.36	0.27
0.19	0.72	0.75	0.08	0.54	0.02	0.01	0.36	0.16	0.28
0.18	0.01	0.95	0.69	0.18	0.47	0.23	0.32	0.82	0.53
0.31	0.42	0.73	0.04	0.83	0.45	0.13	0.57	0.63	0.29

Test for Random Numbers (cont.)

The sequence of runs up and down is as follows:

+ + + − + − + − − − + + − + − − + − + − − + − − + − + + − − + + − + − − + + −

There are 26 runs in this sequence. With N=40 and a=26,

$\mu_a = \{2(40) - 1\} / 3 = 26.33$ and

$\sigma_a^2 = \{16(40) - 29\} / 90 = 6.79$

Then,

$Z_0 = (26 - 26.33) / \sqrt{(6.79)} = -0.13$

Now, the critical value is $Z_{0.025} = 1.96$, so the independence of the numbers cannot be rejected on the basis of this test.

Test for Random Numbers (cont.)

- *Poker Test* - based on the frequency with which certain digits are repeated.

  Example:

  0.255 0.577 0.331 0.414 0.828 0.909

  Note: a pair of like digits appear in each number generated.

Test for Random Numbers (cont.)

In 3-digit numbers, there are only 3 possibilities.

P(3 different digits) =

(2nd diff. from 1st) * P(3rd diff. from 1st & 2nd)

= (0.9) (0.8) = 0.72

P(3 like digits) =

(2nd digit same as 1st) * P(3rd digit same as 1st)

= (0.1) (0.1) = 0.01

P(exactly one pair) = 1 - 0.72 - 0.01 = 0.27

Test for Random Numbers (cont.)

(Example)

A sequence of 1000 three-digit numbers has been generated and an analysis indicates that 680 have three different digits, 289 contain exactly one pair of like digits, and 31 contain three like digits. Based on the poker test, are these numbers independent?

Let $\alpha = 0.05$.

The test is summarized in next table.

Test for Random Numbers (cont.)

Combination, i	Observed Frequency, O_i	Expected Frequency, E_i	$\frac{(O_i - E_i)^2}{E_i}$
Three different digits	680	720	2.24
Three like digits	31	10	44.10
Exactly one pair	289	270	1.33
	1000	1000	47.65

The appropriate degrees of freedom are one less than the number of class intervals. Since $\chi^2_{0.05,\,2} = 5.99 < 47.65$, the independence of the numbers is rejected on the basis of this test.

Perspective

Diehard Tests (George Marsaglia)

1. Birthday spacing
2. Overlapping permutations
3. Rank of matrices
4. Monkey tests
5. Count the 1s
6. Parking lot test
7. Minimum distance test
8. Random spheres test
9. The squeeze test
10. Overlapping sums test
11. Runs test
12. The craps test

DieHarder

Random number generator in stochastic simulations in physics/biology/chemistry: Mersenne Twister

Practical Exercise

- Generate random numbers using a linear congruential generator
- Determine the frequencies of the (N-1) runs up and runs down in the sequence

References

- Simulation and Modeling Analysis, Law & Kelton (1991)
- CSE 808 Modeling and Discrete Simulation- H. Hughes, Carnegie Mellon Univ.